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A CHARACTERISATION OF THE $\mathbf{n}\langle\mathbf{1}\rangle \oplus \langle\mathbf{3}\rangle$ FORM AND APPLICATIONS TO RATIONAL HOMOLOGY SPHERES

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Abstract. We conjecture two generalisations of Elkies' theorem on unimodular quadratic forms to non-unimodular forms. We give some evidence for these conjectures including a result for determinant 3. These conjectures, when combined with results of Frøyshov and of Ozsváth and Szabó, would give a simple test of whether a rational homology 3-sphere may bound a negative-definite four-manifold. We verify some predictions using Donaldson's theorem. Based on this we compute the four-ball genus of some Montesinos knots.

1. Introduction

Let Y be a rational homology three-sphere and X a smooth negative-definite four-manifold bounded by Y. For any Spin^c structure $\mathfrak{t}$ on Y let $d(Y,\mathfrak{t})$ denote the correction term invariant of Ozsváth and Szabó [10]. It is shown in [10, Theorem 9.6] that for each Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$,

$$c_1(\mathfrak{s})^2 + \mathrm{rk}(H^2(X;\mathbb{Z})) \leq 4d(Y,\mathfrak{s}|_Y). \tag{1}$$

This is analogous to a gauge-theoretic result of Frøyshov [5]. These theorems constrain the possible intersection forms that Y may bound. The above inequality is used in [8] to constrain intersection forms of a given rank bounded by Seifert fibred spaces, with application to four-ball genus of Montesinos links. In this paper we attempt to get constraints by finding a lower bound on the left-hand side of (1) which applies to forms of any rank. This has been done for unimodular forms by Elkies:

Theorem 1.1 ([2]). *Let Q be a negative-definite unimodular integral quadratic form of rank n. Then there exists a characteristic vector x with $Q(x,x)+n \geq 0$; moreover, x can be chosen so that the inequality is strict, unless $Q = n\langle -1\rangle$.*

Together with (1) this implies that an integer homology sphere Y with $d(Y) < 0$ cannot bound a negative-definite four-manifold, and if $d(Y) = 0$ then the only definite pairing that Y may bound is the diagonal form. Since $d(S^3) = 0$ this generalises Donaldson's theorem on intersection forms of closed four-manifolds [1].

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In Section 2 we conjecture two generalisations of Elkies' theorem to forms of arbitrary determinant. We prove some special cases, including Theorem 3.1 which is a version of Theorem 1.1 for forms of determinant 3. This implies the following

Theorem 1.2. *Let Y be a rational homology sphere with $H_1(Y;\mathbb{Z}) = \mathbb{Z}/3$ and let $\mathfrak{t}_0$ be the* spin *structure on Y. If Y bounds a negative-definite four-manifold X then either*

$$d(Y, \mathfrak{t}_0) \geq -\frac{1}{2},$$

or

$$\max_{\mathfrak{t} \in \mathrm{Spin}^c(Y)} d(Y, \mathfrak{t}) \geq \frac{1}{6}.$$

If equality holds in both then the intersection form of X is diagonal.

In Section 4 we discuss further topological implications of our conjectures; in particular some predictions for Seifert fibred spaces may be verified using Donaldson's theorem. We find two families of Seifert fibred rational homology spheres, no multiple of which can bound negative-definite manifolds. We use these results to determine the four-ball genus for two families of Montesinos knots, including one whose members are algebraically slice but not slice.

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2. Conjectured generalisations of Elkies' theorem

We begin with some notation. A nondegenerate quadratic form Q of rank n over the integers gives rise to a symmetric matrix with entries $Q(e_i, e_j)$, where $\{e_i\}$ is the standard basis for $\mathbb{Z}^n$; we also denote the matrix by Q. Let Q' denote the induced form on the dual $\mathbb{Z}^n$; this is represented by the inverse matrix. Two matrices Q_1 and Q_2 represent the same form if and only if $Q_1 = P^T Q_2 P$ for some $P \in GL(n, \mathbb{Z})$.

We call $y \in \mathbb{Z}^n$ a *characteristic covector* for Q if

$$y^T \xi \equiv Q(\xi, \xi) \pmod 2 \quad \forall \xi \in \mathbb{Z}^n.$$

We call $x \in \mathbb{Z}^n$ a *characteristic vector* for Q if

$$Q(x, \xi) \equiv Q(\xi, \xi) \pmod 2 \quad \forall \xi \in \mathbb{Z}^n.$$

Note that the form Q induces an injection $x \mapsto Qx$ from $\mathbb{Z}^n$ to its dual with the quotient group having order $|\det Q|$; with respect to the standard bases this map is multiplication by the matrix Q. For unimodular forms this gives a bijection between characteristic vectors and characteristic covectors; in general not every characteristic covector is in the image of the set of characteristic vectors. Also for odd determinant, any two characteristic vectors are congruent modulo 2; this is no longer true for even determinant.

Let Q be a negative-definite integral form of rank n and let δ be the absolute value of its determinant. Denote by $\Delta = \Delta_\delta$ the diagonal form $(n-1)\langle -1 \rangle \oplus$

$\langle -\delta\rangle$. Both of the following give restatements of Theorem 1.1 when restricted to unimodular forms.

Conjecture 2.1. *Every characteristic vector x_0 is congruent modulo 2 to a vector x with*

$$Q(x,x) + n \geq 1 - \delta;$$

moreover, x can be chosen so that the inequality is strict, unless $Q = \Delta_\delta$.

Conjecture 2.2. *There exists a characteristic covector y with*

$$Q'(y,y) + n \geq \begin{cases} 1 - 1/\delta & \text{if } \delta \text{ is odd,} \\ 1 & \text{if } \delta \text{ is even;} \end{cases}$$

moreover, y can be chosen so that the inequality is strict, unless $Q = \Delta_\delta$.

We will discuss the implications of these conjectures in Section 4.

Proposition 2.3. *Conjecture 2.1 is true when restricted to forms of rank ≤ 3, and Conjecture 2.2 is true when restricted to forms of rank 2 and odd determinant.*

Proof. We will first establish Conjecture 2.1 for rank 2 forms. In fact we prove the following stronger statement: if Q is a negative-definite form of rank 2 and determinant δ, then for any $x_0 \in \mathbb{Z}^2$,

$$\max_{x \equiv x_0 (2)} Q(x,x) \geq -1 - \delta, \tag{2}$$

and the inequality is strict unless $Q = \Delta$.

Every negative-definite rank 2 form is represented by a *reduced* matrix

$$Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

with $0 \geq 2b \geq a \geq c$ and $-1 \geq a$. Any vector x_0 is congruent modulo 2 to one of $(0,0), (1,0), (0,1), (1,-1)$; all of these satisy $x^T Q x \geq a + c - 2b$. Thus it suffices to show

$$a + c - 2b \geq -1 - \delta. \tag{3}$$

Note that equality holds in (3) if $Q = \Delta$. Suppose now that $Q \neq \Delta$. Let $Q_\tau = \begin{pmatrix} a + 2\tau & b + \tau \\ b + \tau & c \end{pmatrix}$, and let $\delta_\tau = \det Q_\tau$. Then $a_\tau + c_\tau - 2b_\tau$ is constant and δ_τ is a strictly decreasing function of τ. Thus (3) will hold for Q if it holds for Q_τ for some $\tau > 0$. In the same way we may increase both b and c so that $a + c - 2b$ remains constant and the determinant decreases, or we may increase a and decrease c. In this way we can find a path Q_τ in the space of reduced matrices from any given Q to a diagonal matrix $\begin{pmatrix} -1 & 0 \\ 0 & -\tilde{\delta} \end{pmatrix}$, such that $a + b - 2c$ is constant along the path and the determinant decreases. It follows that (3) holds for Q, and the inequality is strict unless $Q = \Delta$.

A similar but more involved argument establishes Conjecture 2.1 for rank 3 forms. We briefly sketch the argument. Let Q be represented by a reduced matrix of rank 3 (see for example [6]) and let $x_0 \in \mathbb{Z}^3$. By succesively adding 2τ to a diagonal entry and $\pm\tau$ to an off-diagonal entry one may find a path of reduced matrices from Q to $\tilde{Q}$ along which $\max_{x \equiv x_0 (2)} x^T Q x$ is constant and the absolute value of the determinant decreases. One cannot always expect that $\tilde{Q}$ will be diagonal but one can show that the various matrices which arise all satisfy

$$\max_{x \equiv x_0 (2)} x^T \tilde{Q} x \geq -2 - |\det \tilde{Q}|,$$

(with strict inequality unless $\tilde{Q} = \Delta$) from which it follows that this inequality holds for all negative-definite rank 3 forms.

Finally note that for rank 2 forms, the determinant of the adjoint matrix $\operatorname{ad} Q$ is equal to the determinant of Q. Conjecture 2.2 for rank 2 forms of odd determinant now follows by applying (2) to $\operatorname{ad} Q$ and dividing by the determinant δ. □

3. Determinant three

In this section we describe to what extent we can generalise Elkies' proof of Theorem 1.1 to non-unimodular forms. For convenience we work with positive-definite forms. We obtain the following result.

Theorem 3.1. *Let Q be a positive-definite quadratic form over the integers of rank n and determinant 3. Then either Q has a characteristic vector x with $Q(x,x) \leq n+2$ or it has a characteristic covector y with $Q'(y,y) \leq n - \frac{2}{3}$. Moreover, either x or y can be chosen so that the corresponding inequality is strict, unless Q is diagonal.*

Given a positive-definite integral quadratic form Q of rank n, we consider lattices $L \subset L'$ in $\mathbb{R}^n$ (equipped with the standard inner product), with Q the intersection pairing of L, and L' the dual lattice of L. The determinant of the form Q is often referred to as the *discriminant* of the lattice L; however we will use the word determinant in both contexts.

For any lattice $L \subset \mathbb{R}^n$ and a vector $w \in \mathbb{R}^n$ let θ_L^w be the generating function for the norms of vectors in $\frac{w}{2} + L$,

$$\theta_L^w(z) = \sum_{x \in L} e^{i\pi |x + \frac{w}{2}|^2 z};$$

this is a holomorphic function on the upper half-plane $H = \{z \mid \operatorname{Im}(z) > 0\}$. The *theta series* of the lattice L is $\theta_L = \theta_L^0$.

Recall that the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts on H and is generated by S and T, where $S(z) = -\frac{1}{z}$ and $T(z) = z + 1$.

Proposition 3.2. *Let L be an integral lattice of odd determinant δ, and L' its dual lattice. Then*

$$\theta_L(S(z)) = \left(\frac{z}{i}\right)^{n/2} \delta^{-1/2}\theta_{L'}(z) \tag{4}$$

$$\theta_L(TS(z)) = \left(\frac{z}{i}\right)^{n/2} \delta^{-1/2}\theta^w_{L'}(z) \tag{5}$$

$$\theta_{L'}(T^\delta S(z)) = \left(\frac{z}{i}\right)^{n/2} \delta^{1/2}\theta^w_L(z), \tag{6}$$

where w is a characteristic vector in L.

Remark 3.3. *Note that if $w \in L$ is a characteristic vector, then $\theta^w_{L'}$ is a generating function for the squares of characteristic covectors. Under the assumption that the determinant of L is odd, θ^w_L is a generating function for the squares of characteristic vectors.*

Proof. All the formulas follow from Poisson inversion [12, Ch. VII, Proposition 15]. We only need odd determinant in (6). Note that in $\theta_{L'}(z+\delta)$ we can use

$$\delta|y|^2 \equiv |\delta y|^2 \equiv (\delta y, w) \equiv (y, w) \pmod 2 \tag{7}$$

and then apply Poisson inversion. □

Corollary 3.4. *Let L_1 and L_2 be integral lattices of the same rank and the same odd determinant δ. Then*

$$R(z) = \frac{\theta_{L_1}(z)}{\theta_{L_2}(z)}$$

is invariant under T^2 and $ST^{2\delta}S$. Moreover, R^8 is invariant under $(T^2S)^\delta$ and $ST^{\delta-1}ST^{\delta-1}S$.

Proof. Since L is integral, $\theta_L(z+2) = \theta_L(z)$, hence R is T^2 invariant. The squares of vectors in L' belong to $\frac{1}{\delta}\mathbb{Z}$, so $\theta_{L'}(z+2\delta) = \theta_{L'}(z)$. From (4) it follows that $R(S(z)) = \frac{\theta_{L'_1}(z)}{\theta_{L'_2}(z)}$, which gives the $ST^{2\delta}S$ invariance of R.

To derive the remaining symmetries of R^8 we need to use (5) and (6). Let w be a characteristic vector in L. Clearly

$$\delta|y+\frac{w}{2}|^2 = \delta|y|^2 + \delta(y,w) + \frac{\delta}{4}|w|^2$$

holds for any $y \in L'$, so it follows from (7) that

$$\theta^w_{L'}(z+\delta) = e^{i\pi\delta|w|^2/4}\theta^w_{L'}(z).$$

Using (5) we now conclude that R^8 is invariant under $TST^\delta ST^{-1} = (ST^{-2})^\delta$; the last equality follows from the relation $(ST)^3 = 1$ in the modular group. The remaining invariance of R^8 is derived in a similar way from (6). □

From now on we restrict our attention to determinant $\delta = 3$. Consider the subgroup Γ_3 of Γ generated by T^2, ST^6S and ST^2ST^2S. Clearly Γ_3 is a subgroup of $\Gamma_+ = \langle S, T^2\rangle \subset \Gamma$.

Lemma 3.5. *A full set of coset representatives for Γ_3 in Γ_+ is I, S, ST^2, ST^4. Hence a fundamental domain D_3 for the action of Γ_3 on the hyperbolic plane H is the hyperbolic polygon with vertices $-1, -\frac{1}{3}, -\frac{1}{5}, 0, 1, i\infty$.*

Proof. Call $x, y \in \Gamma_+$ *equivalent* if $y = zx$ for some $z \in \Gamma_3$. For an element $x = T^{k_1}ST^{k_2}S \cdots T^{k_n}$ with all $k_i \neq 0$ define the *length* of x, Sx, xS and SxS to be n. Any element $x \in \Gamma_+$ of length $n \geq 2$ is equivalent to one of the form ST^kSy with $k = 0, \pm 2$ and length at most n. If $x = ST^kST^ly$ with $k = \pm 2$ and length $n \geq 2$, then x is equivalent to $ST^{l-k}y$, which has length $\leq n-1$. It follows by induction on length that any element of Γ_+ is equivalent to one with length at most 1. Moreover, if the element has length 1, it is equivalent to ST^k, $k = 2, 4$.

Finally, recall that a fundamental domain for Γ_+ is $D_+ = \{z \in H \mid -1 \leq \mathrm{Re}(z) \leq 1,\ |z| \geq 1\}$ so we can take D_3 to be the union of D_+ and $S(D_+ \cup T^2(D_+) \cup T^4(D_+))$. □

Proof of Theorem 3.1. Suppose that L is a lattice of determinant 3 and rank n for which the square of any characteristic vector is at least $n+2$ and the square of any characteristic covector is at least $n - \frac{2}{3}$. Let Δ be the lattice with intersection form $(n-1)\langle 1\rangle \oplus \langle 3\rangle$; recall from [2] that θ_Δ does not vanish on H. Then

$$R(z) = \frac{\theta_L(z)}{\theta_\Delta(z)}$$

is holomorphic on H and it follows from Corollary 3.4 that R^8 is invariant under Γ_3. We want to show that R is bounded. We will use the following identities that follow from Proposition 3.2:

$$R(S(z)) = \frac{\theta_{L'}(z)}{\theta_{\Delta'}(z)}, \quad R(TS(z)) = \frac{\theta^w_{L'}(z)}{\theta^w_{\Delta'}(z)}, \quad R(ST^\delta S(z)) = \frac{\theta^w_L(z)}{\theta^w_\Delta(z)}.$$

Since the theta series of any lattice converges to 1 as $z \to i\infty$, $R(z) \to 1$ as $z \to 0, i\infty$. By assumption the square of any characteristic covector for L is at least as large as the square of the shortest characteristic covector for Δ. Since the asymptotic behaviour as $z \to i\infty$ of the generating function for the squares of characteristic covectors is determined by the smallest square, it follows from the middle expression for R above that $R(z)$ is bounded as $z \to 1$. Similarly, using the condition on characteristic vectors and the right-most expression for R as $z \to i\infty$, it follows that $R(z)$ is bounded as $z \to -\frac{1}{3}$. Note that $T^{-2}(1) = -1$ and $ST^6S(1) = -\frac{1}{5}$, so $R(z)$ is also bounded as $z \to -1, -\frac{1}{5}$.

Let f be the function on $\Sigma = H/\Gamma_3$ induced by R^8. Then f is holomorphic and bounded, so it extends to a holomorphic function on the compactification of Σ. It follows that $R(z) = 1$, so the theta series of L is equal to the theta series of Δ. Then L contains $n-1$ pairwise orthogonal vectors of square 1, so its intersection form is $(n-1)\langle 1\rangle \oplus \langle 3\rangle$. □

4. Applications

In this section we consider applications to rational homology spheres and the four-ball genus of knots. We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose that $Y = \partial X$ and that Q is the intersection form on $H_2(X;\mathbb{Z})$. Then Q is a quadratic form of determinant ± 3. For any $\mathfrak{s} \in \mathrm{Spin}^c(X)$, let $c(\mathfrak{s})$ denote the image of the first Chern class $c_1(\mathfrak{s})$ modulo torsion. Then $c(\mathfrak{s})$ is a characteristic covector for Q; moreover if $\mathfrak{s}|_Y$ is spin then $c(\mathfrak{s})$ is Qx for some characteristic vector x. The result now follows from Theorem 3.1 and (1). □

Conjectures 2.1 and 2.2 imply the following more general statement.

Conjecture 4.1. *Let Y be a rational homology sphere with $|H_1(Y;\mathbb{Z})| = h$. If Y bounds a negative-definite four-manifold X with no torsion in $H_1(X;\mathbb{Z})$ then*

$$\min_{\mathfrak{t}_0 \in \mathrm{Spin}(Y)} d(Y,\mathfrak{t}_0) \geq (1-h)/4,$$

and

$$\max_{\mathfrak{t} \in \mathrm{Spin}^c(Y)} d(Y,\mathfrak{t}) \geq \begin{cases} \left(1-\dfrac{1}{h}\right)/4 & \text{if } h \text{ is odd,} \\ 1/4 & \text{if } h \text{ is even.} \end{cases}$$

If equality holds in either inequality the intersection form of X is Δ_h.

More generally if Y bounds X with torsion in $H_1(X;\mathbb{Z})$, the absolute value of the determinant of the intersection pairing of X divides h with quotient a square (see for example [8, Lemma 2.1]). One may then deduce inequalities as above corresponding to each choice of determinant; care must be taken since for example not all spin structures on Y extend to spin^c structures on X.

Remark 4.2. *Given a rational homology sphere Y bounding X with no torsion in $H_1(X;\mathbb{Z})$, the intersection pairing of X gives a presentation matrix for $H^2(Y;\mathbb{Z})$ (and also determines the linking pairing of Y). There should be analogues of Conjectures 2.1 and 2.2 which restrict to forms presenting a given group (and inducing a given linking pairing). These should give stronger bounds than those in Conjecture 4.1.*

4.1. Seifert fibred examples. In Examples 4.5 and 4.6 we list families of Seifert fibred spaces Y which bound positive-definite but not negative-definite four-manifolds. It follows as in [4, Theorem 10.2] that for any $m > 0$, the connected sum of m copies of Y cannot bound a negative-definite four-manifold. In Examples 4.7 through 4.9 we list families of Seifert fibred spaces which can only bound the diagonal negative-definite form Δ_δ (or sometimes Δ_1). We found these examples using predictions based on Conjecture 4.1 and verified them using Donaldson's theorem via Proposition 4.4. Finally, in Example 4.10 we exhibit a family of Seifert fibred spaces which according to the conjecture can only bound Δ_δ. For this family the method of Proposition 4.4 does not apply.

In what follows we extend the definition of Δ_1 to include the trivial form on the trivial lattice. Also note that a lattice uniquely determines a quadratic form, and a form determines an equivalence class of lattices; in the rest of this section we use the terms lattice and form interchangeably.

Definition 4.3. *Let L be a lattice of rank m and determinant δ. We say L is* rigid *if any embedding of L in $\mathbb{Z}^n$ is contained in a $\mathbb{Z}^m$ sublattice. We say L is* almost-rigid *if any embedding of L in $\mathbb{Z}^n$ is either contained in a $\mathbb{Z}^m$ sublattice, or contained in a $\mathbb{Z}^{m+1}$ sublattice with orthogonal complement spanned by a vector v with $|v|^2 = \delta$.*

Proposition 4.4. *Let Y be a rational homology sphere and let h be the order of $H_1(Y;\mathbb{Z})$. Suppose Y bounds a positive-definite four-manifold X_1 with $H_1(X_1;\mathbb{Z}) = 0$. Let Q_1 be the intersection pairing of X_1 and let m denote its rank.*

If Q_1 does not embed into $\mathbb{Z}^n$ for any n then Y cannot bound a negative-definite four-manifold.

If Q_1 is rigid and Y bounds a negative-definite X_2 then h is a square and $Q_2 = \Delta_1$; if $h > 1$, then there is torsion in $H_1(X_2;\mathbb{Z})$.

If Q_1 is almost-rigid and Y bounds a negative-definite X_2 then either

- *$Q_2 = \Delta_h$ or*
- *Q_1 embeds in $\mathbb{Z}^m$, h is a square and $Q_2 = \Delta_1$; if $h > 1$, then there is torsion in $H_1(X_2;\mathbb{Z})$.*

Proof. Suppose Y bounds a negative-definite X_2 with intersection pairing Q_2. Then $X = X_1 \cup_Y -X_2$ is a closed positive-definite manifold. The Mayer-Vietoris sequence for homology and Donaldson's theorem yield an embedding $\iota : Q_1 \oplus -Q_2 \to \mathbb{Z}^{m+k}$, where k is the rank of Q_2.

If the image of Q_1 under ι is contained in a $\mathbb{Z}^m$ sublattice, then the image of $-Q_2$ is contained in the orthogonal $\mathbb{Z}^k$ sublattice. Now consider the Mayer-Vietoris sequence for cohomology:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^2(X;\mathbb{Z}) & \longrightarrow & H^2(X_1;\mathbb{Z}) & \oplus & H^2(X_2;\mathbb{Z}) & \longrightarrow & H^2(Y;\mathbb{Z}), \\ & & & \| & & \| & & \\ & & & Q_1' & & -Q_2' \oplus T_2 & & \end{array}$$

where T_2 is the torsion subgroup and Q' denotes the dual lattice to Q. This yields an embedding $\iota' : \mathbb{Z}^{m+k} \to Q_1' \oplus -Q_2'$. The mapping ι' is hom-dual to ι and hence also decomposes orthogonally, sending $\mathbb{Z}^m$ to Q_1' and $\mathbb{Z}^k$ to $-Q_2'$. The image of $\mathbb{Z}^m$ in Q_1' has index $\sqrt{h}$, since h is the determinant of Q_1. (In general if $L_1 \subset L_2$ are lattices of the same rank then the square of the index $[L_2 : L_1]$ is the quotient of their determinants.) The restriction map from $H^2(X_1;\mathbb{Z})$ to $H^2(Y;\mathbb{Z})$ is onto, so its kernel K is a subgroup of $\mathbb{Z}^m$ of index $\sqrt{h}$. It follows that $\mathbb{Z}^m/K$ injects into T_2 and that the image of T_2 in $H^2(Y;\mathbb{Z})$ has order $t \geq \sqrt{h}$. Then by [8, Lemma 2.1], $t = \sqrt{h}$ and Q_2 is unimodular. Since $-Q_2$ is a sublattice of $\mathbb{Z}^k$ we have $Q_2 = \Delta_1$.

Suppose now that the image of Q_1 under ι is contained in a $\mathbb{Z}^{m+1}$ sublattice, and its orthogonal complement in $\mathbb{Z}^{m+1}$ is spanned by a vector v with $|v|^2 = h$. Then the image of $-Q_2$ is a sublattice of $(k-1)\langle 1\rangle \oplus \langle h\rangle$; it therefore has determinant at least h. On the other hand its determinant divides h [8, Lemma 2.1]. It follows that Q_2 is equal to Δ_h. □

If Y is the Seifert fibred space $Y(e;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$, let

$$k(Y) = e\alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3.$$

If $k(Y) \neq 0$ then Y is a rational homology sphere and $|k(Y)|$ is the order of $H_1(Y;\mathbb{Z})$. Furthermore, if $k(Y) < 0$ then Y bounds a positive-definite plumbing. For our conventions for lens spaces and Seifert fibred spaces see [8]. Recall in particular that (α_i,β_i) are coprime pairs of integers with $\alpha_i \geq 2$. We will also assume here that $1 \leq \beta_i < \alpha_i$.

Example 4.5. *Seifert fibred spaces $Y = Y(-2;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$ with*

$$\frac{\alpha_1}{\beta_1} \leq 2, \quad \frac{\alpha_2}{\beta_2}, \frac{\alpha_3}{\beta_3} < 2, \quad k(Y) < 0,$$

cannot bound negative-definite four-manifolds.

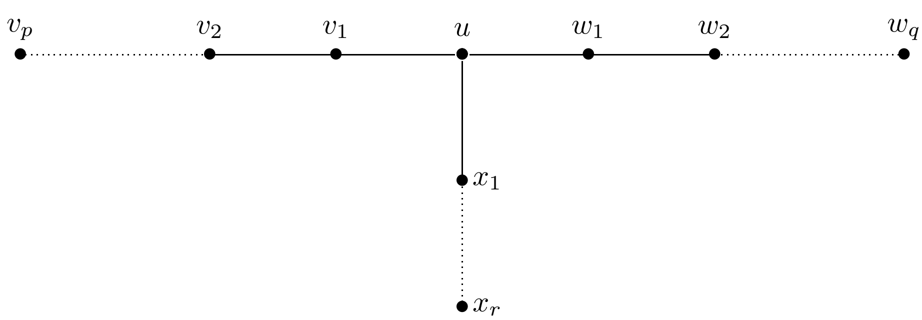

FIGURE 1. **Plumbing graph.**

Proof. Note that Y is the boundary of the positive-definite plumbing shown in Figure 1, where vertices u, v_1, w_1 and x_1 have square 2 and v_2 and w_2 have square at least 2. This lattice does not admit an embedding in any $\mathbb{Z}^n$. To see this let $e_1,\ldots,e_n$ be the standard basis of $\mathbb{Z}^n$. The vertex u must map to an element of square 2, which we may suppose is e_1+e_2. The 3 adjacent vertices must be mapped to elements of the form e_1+e_3, e_1-e_3 and e_2+e_4. Now we see that it is not possible to map the remaining 2 vertices v_2 and w_2; we are only able to further extend the map along the leg of the graph emanating from the vertex mapped to e_2+e_4. □

Example 4.6. *Seifert fibred spaces $Y = Y(-2; (\alpha_1, \beta_1), (\alpha_2, \alpha_2-1), (\alpha_3, \alpha_3-1))$ with*

$$\alpha_2, \alpha_3 \geq \frac{\alpha_1}{\beta_1}, \quad \alpha_3 \geq 3, \quad k(Y) < 0,$$

cannot bound negative-definite four-manifolds unless

$$\beta_1 = 1, \quad \min(\alpha_2, \alpha_3) = \alpha_1.$$

In the latter case, if Y bounds a negative-definite X then the intersection pairing of X is Δ_1 and the torsion subgroup of $H_1(X;\mathbb{Z})$ is nontrivial.

Proof. In this case Y is again the boundary of a positive-definite plumbing as in Figure 1. The vertices u, v_i and w_j have square 2, and $p = \alpha_2 - 1$, $q = \alpha_3 - 1$. Vertex x_1 has square $a = \lceil \frac{\alpha_1}{\beta_1} \rceil$. If $\frac{\alpha_1}{\beta_1} = \min(\alpha_2, \alpha_3) = a$ then by inspection this pairing is rigid with determinant $a^2 > 1$; otherwise it does not admit any embedding into $\mathbb{Z}^n$. For more details see the proof of Example 4.8. □

Example 4.7. *The only negative-definite pairing that $L(p, 1)$ can bound is the diagonal form Δ_p unless $p = 4$ in which case it may also bound Δ_1. (Note that $L(p, 1)$ is the boundary of the disk bundle over S^2 with intersection pairing $\langle -p \rangle$.)*

Proof. By A_m we denote the plumbing according to a linear graph with m vertices whose weights are 2. Observe that $L(p, 1)$ is the boundary of the positive-definite plumbing A_{p-1}. If $p \neq 4$ then up to automorphisms of $\mathbb{Z}^n$ there is a unique embedding of A_{p-1} in $\mathbb{Z}^n$; the image is contained in a $\mathbb{Z}^p$ and its orthogonal complement in $\mathbb{Z}^p$ is generated by the vector $(1, 1, \ldots, 1)$. Hence A_{p-1} is almost-rigid and does not embed in $\mathbb{Z}^{p-1}$. However, A_3 also admits an embedding in $\mathbb{Z}^3$. □

Example 4.8. *If $Y = Y(-2; (\alpha_2\beta_1 + 1, \beta_1), (\alpha_2, \alpha_2 - 1), (\alpha_3, \alpha_3 - 1))$ with $\alpha_3 > \alpha_2$, then the only negative-definite pairing that Y may bound is the diagonal form $\Delta_{|k(Y)|}$ unless*

$$\beta_1 = 1, \quad \alpha_3 = \alpha_2 + 1.$$

In the latter case the only negative-definite pairings that Y may bound are $\Delta_{|k(Y)|}$ and Δ_1.

Proof. Note this is a borderline case of Example 4.6. In the notation of that example $\alpha_2 = a-1$. The positive-definite plumbing is similar to that in Example 4.6 with $r = \beta_1$; also the vertices x_l with $l > 1$ all have square 2. Denote the pairing associated to this plumbing by Q. We consider an embedding of Q into $\mathbb{Z}^n$. Let e_i, f_j and g_l denote unit vectors in $\mathbb{Z}^n$. Without loss of generality the vertex u maps to $e_1 + f_1$. Then v_i maps to $e_{i-1} + e_i$ and w_j maps to $f_{j-1} + f_j$.

Now consider the image of x_1. This may map to $e_1 - e_2 + \cdots \pm e_{a-1} + g_1$; then x_l maps to $g_{l-1} + g_l$ for $l > 1$. Thus the image of Q is contained in a $\mathbb{Z}^{p+q+r+2}$ sublattice. The determinant of Q is $|k(Y)| = {\alpha_2}^2\beta_1 + \alpha_2 + \alpha_3$ (note $k(Y) < 0$). The orthogonal complement of Q in $\mathbb{Z}^{p+q+r+2}$ is spanned by the vector $\sum(-1)^{i-1}e_i + \sum(-1)^j f_j + \alpha_2 \sum(-1)^l g_l$, whose square is $|k(Y)|$. Up to automorphism this is the only embedding of Q into $\mathbb{Z}^n$ unless $\alpha_3 = a$ and $\beta_1 = 1$. In this case x_1 may map to the alternating sum $f_1 - f_2 + \cdots \pm f_a$; the image of the resulting embedding is contained in $\mathbb{Z}^{p+q+r+1}$. □

Example 4.9. *If $Y = Y(-1; (3,1), (3a+1, a), (5b+3, 2b+1))$ with $k(Y) < 0$, then the only negative-definite pairing that Y may bound is the diagonal form $\Delta_{|k(Y)|}$ unless $a = b = 1$ in which case it may also bound Δ_1.*

Proof. Note that the condition $k(Y) < 0$ implies $a = 1$ or $b = 0$ or $a = b+1 = 2$. Again, Y is the boundary of a positive-definite plumbing as in Figure 1, with $p = a$, $q = b+1$ and $r = 1$. The vertex u has square 1, w_1 and x_1 have square 3, v_1 has square 4. If $a > 1$ then v_j has square 2 for $j > 1$. If $b > 0$ then w_2 has square 3, and any remaining w_i has square 2. Denote the pairing associated to this plumbing by Q. We consider an embedding of Q into $\mathbb{Z}^n$. Let e_i denote unit vectors in $\mathbb{Z}^n$. Without loss of generality the vertex u maps to e_1, x_1 maps to $e_1 + e_2 + e_3$ and w_1 maps to $e_1 - e_2 + e_4$. Then v_1 has to map to $e_1 - e_3 - e_4 + e_5$. Now w_2, if present, has to map to $e_4 + e_5 + e_6$ or $-e_2 + e_3 + e_5$; the second possibility only works if $a = b = 1$. Finally v_2, if present, has to map to $e_5 - e_6$. The reader may verify that Q is almost-rigid. □

Example 4.10. *Let $Y_a = Y(-2; (2,1), (3,2), (a, a-1))$ with $a \geq 7$. Then $h = k(Y) = a - 6$,*

$$\min_{\mathfrak{t}_0 \in \mathrm{Spin}(Y)} d(Y, \mathfrak{t}_0) = (1-h)/4$$

and

$$\max_{\mathfrak{t} \in \mathrm{Spin}^c(Y)} d(Y, \mathfrak{t}) = \begin{cases} \left(1 - \dfrac{1}{h}\right)/4 & \text{if } h \text{ is odd,} \\ 1/4 & \text{if } h \text{ is even.} \end{cases}$$

If a is 7 or 9 then the only negative-definite form Y_a bounds is Δ_h. If Conjecture 4.1 holds then the same is true for all Y_a.

Proof. Y_a is the boundary of the negative-definite plumbing with intersection pairing given by

$$Q = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -a \end{pmatrix},$$

which represents $3\langle -1 \rangle \oplus \langle -a+6 \rangle$. The computations of $d(Y)$ follow as in [11]. The claim for Y_7 follows from the discussion following Theorem 1.1. The claim for Y_9 follows from Theorem 1.2. □

4.2. Four-ball genus of Montesinos knots.

Let K be a knot in S^3 and let g denote its Seifert genus. The four-ball genus g^* of K is the minimal genus of a smooth surface in B^4 with boundary K. A classical result of Murasugi states that $g^* \geq |\sigma|/2$, where σ is the signature of K. If this lower bound is attained then the double branched cover of S^3 along K bounds a definite four-manifold with signature σ. The double branched cover of the Montesinos knot or link $M(e;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$ is $Y(-e;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$. (For more details see [8].)

The following generalises an example of Fintushel and Stern [4].

Example 4.11. *The pretzel knot* $K(p,-q,-r) = M(2;(p,1),(q,q-1),(r,r-1))$ *for odd* p*,* q *and* r *satisfying*

$$q,r > p > 0 \quad \textit{and} \quad pq+pr-qr \textit{ is a square}$$

is algebraically slice but has $g^* = 1$*.*

Proof. The knot has a genus 1 Seifert surface yielding the Seifert matrix

$$M = \begin{pmatrix} \frac{p-r}{2} & \frac{p+1}{2} \\ \frac{p-1}{2} & \frac{p-q}{2} \end{pmatrix}.$$

The vector $x = (p-l, r-p)$, where $l = \sqrt{pq+pr-qr}$, satisfies $x^T M x = 0$, demonstrating the knot is algebraically slice. The double branched cover Y of the knot has $k(Y) = -l^2$. From Example 4.6 we see that Y does not bound a rational homology ball. It follows that $0 < g^* \leq g = 1$.

It is shown by Livingston [7] that $K(p,-q,-r)$ has $\tau = 1$, where τ is the Ozsváth-Szabó knot concordance invariant. This also gives $g^* = 1$. □

In the following example $m(K)$ refers to a knot invariant due to Taylor (see for example [8]). This is computable from any Seifert matrix for K and satisfies the inequalities

$$g^* \geq m \geq |\sigma|/2.$$

Example 4.12. *The Montesinos knot* $K_{q,r} = M(2;(qr-1,q),(r+1,r),(r+1,r))$ *with odd* $q \geq 3$ *and even* $r \geq 2$*, has signature* $\sigma = 1-q$ *and has*

$$g = g^* = \frac{q+1}{2}.$$

Computations suggest that Taylor's invariant $m(K_{q,r})$ *is* $\frac{q-1}{2}$*.*

Proof. The knot $K_{q,r}$ is equal to $M(0;(qr-1,q),(r+1,-1),(r+1,-1))$. It is easily seen that $K_{q,r}$ has a spanning surface with genus $\frac{q+1}{2}$. Using the resulting Seifert matrix one gets the formula for the signature. The double branched cover Y of $K_{q,r}$ has $k(Y) < 0$. From Example 4.6 we see that Y does not bound a negative-definite four-manifold; the genus formula follows.

We have computed $m(K_{q,r})$ for $q < 10000$ and any r. □

Remark 4.13. *We have discussed Conjectures 2.1 and 2.2 with Noam Elkies. He has suggested an alternative proof of Theorem 3.1 using gluing of lattices* [3]*. His proof works for odd determinants* δ *up to 11, under the additional assumption that there is an element of* L' *whose square is congruent to* $1/\delta$ *modulo 1.*

A proof of Conjecture 2.2, using gluing of lattices, will appear in [9]*.*

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