

BEST UNIFORM RATIONAL APPROXIMATION OF x^α ON $[0, 1]$

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ABSTRACT. A strong error estimate for the uniform rational approximation of x^α on $[0, 1]$ is given, and its proof is sketched. Let $E_{nn}(x^\alpha, [0, 1])$ denote the minimal approximation error in the uniform norm. Then it is shown that

$$\lim_{n \rightarrow \infty} e^{2\pi\sqrt{\alpha n}} E_{nn}(x^\alpha, [0, 1]) = 4^{1+\alpha} |\sin \pi\alpha|$$

holds true for each $\alpha > 0$.

1. INTRODUCTION

Let Π_n denote the set of all polynomials of degree at most $n \in \mathbb{N}$ with real coefficients; \mathcal{R}_{mn} denote the set $\{p/q | p \in \Pi_m, q \in \Pi_n, q \neq 0\}$, $m, n \in \mathbb{N}$, of rational functions; and the *best rational approximant* $r_{mn}^* \in \mathcal{R}_{mn}$, $m, n \in \mathbb{N}$, and the *minimal approximation error* $E_{mn} = E_{mn}(x^\alpha, [0, 1])$ be defined by

$$(1.1) \quad E_{mn} := \|x^\alpha - r_{mn}^*\|_{[0,1]} = \inf_{r \in \mathcal{R}_{mn}} \|x^\alpha - r\|_{[0,1]},$$

where $\|\cdot\|_K$ denotes the sup norm on $K \subseteq \mathbb{R}$. It is well known that the best approximant r_{mn}^* exists and is unique within \mathcal{R}_{mn} (cf. [Me, §§9.1, 9.2] or [Ri, §5.1]). The unique existence also holds in the special case ($n = 0$) of *best polynomial approximants*.

Since $f_\alpha(x) := |x|^\alpha$ is an even function on $[-1, 1]$, the same is true for its unique approximant $r_{mn}^* = r_{mn}^*(f_\alpha, [-1, 1]; \cdot)$, and consequently a substitution of z^2 by z shows that approximating $|x|^{2\alpha}$ on $[-1, 1]$ and x^α on $[0, 1]$ poses an identical problem. We have

$$(1.2) \quad E_{2m,2n}(|x|^{2\alpha}, [-1, 1]) = E_{mn}(x^\alpha, [0, 1]) \quad \text{for all } m, n \in \mathbb{N}.$$

From Jackson's and Bernstein's theorems about the interdependence of approximation speed and smoothness of the function to be approximated (cf. [Me, §§5.5, 5.6]) we know that in case of $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ the minimal error $E_{m,0}(|x|^\alpha, [-1, 1])$ behaves like $O(m^{-\alpha})$ as $m \rightarrow \infty$. In [Be1, Be2] Bernstein proved a result that is deeper and much more difficult to obtain; he showed that the limit

$$(1.3) \quad \lim_{m \rightarrow \infty} m^\alpha E_{m,0}(|x|^\alpha, [-1, 1]) := \beta(\alpha)$$

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exists for each $\alpha > 0$; however, an explicit expression for the constant $\beta(\alpha)$, $\alpha > 0$, still is not known.

In case $\alpha = 1$ the number $\beta := \beta(1) = 0.28016\dots$ is known as *Bernstein's constant*. In [Bel, p. 56] Bernstein raised the question whether β can be expressed by known transcendentals or whether it defines a new one, and based on numerical upper and lower bounds for β , which he calculated up to a precision of ± 0.005 , he made the tentative conjecture $\beta \stackrel{?}{=} 1/(2\sqrt{\pi})$, which, however, has been disproved in [VC1] by extensive and nontrivial high precision calculations.

For large values of α , Bernstein was able to establish in [Be2] an asymptotic expression. He showed that

$$(1.4) \quad \lim_{\alpha \rightarrow \infty} \frac{\beta(\alpha)}{\Gamma(\alpha) |\sin(\pi\alpha/2)|} = \frac{1}{\pi}.$$

While Bernstein's investigations on best polynomial approximation of $|x|$ and $|x|^\alpha$ were published in the period between 1909 and 1938, the study of best rational approximation of $|x|^\alpha$ was only started in 1964 by Newman's surprising (at the time) result in [Ne] that

$$(1.5) \quad \frac{1}{2}e^{-9\sqrt{n}} \leq E_{nn}(|x|, [-1, 1]) \leq 3e^{-\sqrt{n}} \quad \text{for all } n = 4, 5, \dots$$

A comparison of this result with (1.3) shows that the convergence behavior of rational approximants is essentially better than that of polynomials. Newmann's investigation has triggered a whole series of contributions, from which we select a short list with papers that contain substantial improvements of the error estimate in the uniform norm.

$$\begin{aligned} E_{nn}(x^\alpha, [0, 1]) &\leq e^{-c(\alpha)\sqrt[3]{n}}, & \alpha \in \mathbb{R}_+ & \text{in [FrSz];} \\ E_{nn}(x^{1/3}, [0, 1]) &\leq e^{-c\sqrt{n}}, & & \text{in [Bu1];} \\ E_{nn}(x^\alpha, [0, 1]) &\leq e^{-c(\alpha)\sqrt{n}}, & \alpha \in \mathbb{R}_+, & \text{in [Go1];} \\ \frac{1}{3}e^{-\pi\sqrt{2n}} \leq E_{nn}(x^{1/2}, [0, 1]) &\leq e^{-\pi\sqrt{2n}(1-\mathcal{O}(n^{-1/4}))}, & & \text{in [Bu2];} \\ e^{-c(\alpha)\sqrt{n}} \leq E_{nn}(x^\alpha, [0, 1]), & \alpha \in \mathbb{Q}_+ \setminus \mathbb{N}, & & \text{in [Go2];} \\ e^{-4\pi\sqrt{\alpha n}(1+\varepsilon)} \leq E_{nn}(x^\alpha, [0, 1]) &\leq e^{-\pi\sqrt{\alpha n}(1-\varepsilon)}, & \alpha \in \mathbb{R}_+ \setminus \mathbb{N}, \varepsilon > 0, n \geq n_0(\varepsilon, \alpha) & \text{in [Go3];} \\ E_{nn}(x^{1/2}, [0, 1]) &\leq cne^{-\pi\sqrt{2n}}, & & \text{in [Vj1];} \\ \frac{1}{3}e^{-\pi\sqrt{2n}} \leq E_{nn}(x^{1/2}, [0, 1]) &\leq ce^{-\pi\sqrt{2n}} & & \text{in [Vj2];} \\ e^{-c_1(s)\sqrt{n}} \leq E_{nn}(\sqrt[s]{x}, [0, 1]) &\leq e^{-c_2(s)\sqrt{n}}, & s \in \mathbb{N} & \text{in [Tz].} \end{aligned}$$

Here $c, c(\alpha), \dots$ denote constants. Relation (1.2) allows us to transfer these results to the problem of approximating $|x|^\alpha$ on $[-1, 1]$.

It may be appropriate to repeat a remark from [Go2], where it was pointed out that Newman's result can be obtained rather immediately from an old result (from 1877) by Zolotarev. In this sense, the investigation of rational approximants dates back even further than Bernstein's work on polynomial approximants.

The best result known so far for the rational approximation of x^α on $[0, 1]$ was obtained independently by Ganelius [Ga] in 1979 and by Vjacheslavov [Vj3] in 1980. They proved that for $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ there exists a constant $c_1 = c_1(\alpha) > 0$ such that

$$(1.6) \quad \liminf_{n \rightarrow \infty} e^{2\pi\sqrt{\alpha n}} E_{nn}(x^\alpha, [0, 1]) \geq c_1(\alpha)$$

and, conversely, that for each positive rational number $\alpha \in \mathbb{Q}_+$ there exists a constant $c_2 = c_2(\alpha) < \infty$ such that

$$(1.7) \quad \limsup_{n \rightarrow \infty} e^{2\pi\sqrt{\alpha n}} E_{nn}(x^\alpha, [0, 1]) \leq c_2(\alpha).$$

Both authors were not able to show that $c_2 = c_2(\alpha)$ depends continuously on α . Thus, inequality (1.7) remained open for $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$; however, in [Ga] Ganelius was able to prove the somewhat weaker estimate

$$(1.8) \quad E_{nn}(x^\alpha, [0, 1]) \leq c_2(\alpha) e^{2\pi\sqrt{\alpha n} + c_3(\alpha) \sqrt[4]{n}} \quad \text{for } n \geq n_0(c_2(\alpha), c_3(\alpha)),$$

which holds for each $\alpha > 0$. In (1.8) $c_2(\alpha)$ and $c_3(\alpha)$ are constants depending on α .

The results (1.6)–(1.8) give the correct exponent $-2\pi\sqrt{\alpha n}$ in the error formula; however, nearly nothing is said about the coefficient in front of the exponential term. The determination of this coefficient is the subject of the present note. Practically as a byproduct, we prove the upper estimate (1.7) for irrational exponents $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$.

2. THE RESULT

Theorem 1. *The limit*

$$(2.1) \quad \lim_{n \rightarrow \infty} e^{2\pi\sqrt{\alpha n}} E_{nn}(x^\alpha, [0, 1]) = 4^{1+\alpha} |\sin \pi \alpha|$$

holds for each $\alpha > 0$.

Remarks. (1) From (2.1) we deduce that the approximation error $E_{nn}(x^\alpha, [0, 1])$ has the asymptotic behavior

$$(2.2) \quad E_{nn}(x^\alpha, [0, 1]) = 4^{1+\alpha} |\sin \pi \alpha| e^{-2\pi\sqrt{\alpha n}} (1 + o(1)) \quad \text{as } n \rightarrow \infty,$$

and equivalently it follows with (1.2) that

$$(2.3) \quad E_{nn}(|x|^\alpha, [-1, 1]) = 4^{1+\alpha/2} |\sin \pi \alpha / 2| e^{-\pi\sqrt{\alpha n}} (1 + o(1)) \quad \text{as } n \rightarrow \infty$$

for each $\alpha > 0$.

(2) Not only the explicit expression on the right-hand side of (2.1) but already the existence of the limit represents a result difficult to obtain. The value $4^{1+\alpha} |\sin \pi \alpha|$, $\alpha > 0$, is the analogue of Bernstein's constant $\beta(\alpha)$ in (1.3) for the case of rational approximation. It has already been noted in the introduction that an explicit expression for $\beta(\alpha)$ is still not known. The best we know is Bernstein's asymptotic formula (1.4). Since in (1.4) we have considered approximation on $[-1, 1]$, the counterpart of the asymptotic value $\frac{1}{\pi} \Gamma(\alpha) |\sin \pi \alpha / 2|$ for $\beta(\alpha)$ is the value $4^{1+\alpha/2} |\sin(\pi \alpha / 2)|$ in case of rational approximation.

(3) If we turn our attention to the special case of rational approximation of $|x|$ on $[-1, 1]$, then it follows from (1.2) that

$$(2.4) \quad E_{2n,2n}(|x|, [-1, 1]) = E_{nn}(\sqrt{x}, [0, 1]) \quad \text{for } n \in \mathbb{N},$$

and hence we deduce from (2.1) that

$$(2.5) \quad \lim_{n \rightarrow \infty} e^{-\pi\sqrt{n}} E_{nn}(|x|, [-1, 1]) = 4^{1+1/2} |\sin \frac{\pi}{2}| = 8.$$

Limit (2.5) has been conjectured in [VRC] on the basis of high precision calculations and was proved in [St].

It may be surprising that in case of rational approximation, which is in many respects more complex than the polynomial case, limit (2.5) has a rational value, while in the polynomial case Bernstein's question in [Be1] about the character of the number $\beta = \beta(1)$ is still open and the numerical results in [VC1] show that β cannot be a rational number with a moderately small denominator.

3. OUTLINE OF THE PROOF OF THEOREM 1

If limit (2.1) is proved for one of the paradiagonal sequences

$$\{E_{n+k,n}(x^\alpha, [0, 1])\}_{n=|k|}^\infty,$$

$k \in \mathbb{Z}$ fixed, then it holds also for the diagonal sequence

$$\{E_{nn}(x^\alpha, [0, 1])\}_{n=1}^\infty.$$

It turns out that

$$(3.1) \quad n + k = m := n + 1 + [\alpha], \quad \alpha \in \mathbb{R}_+ \setminus \mathbb{N}, n \in \mathbb{N},$$

is a good choice for the numerator degree m . From the theory of best rational approximants we learn that the error function

$$(3.2) \quad e_n(z) := z^\alpha - r_{mn}^*(z), \quad z \in \mathbb{C} \setminus \mathbb{R}_-,$$

has exactly $2n + 2 + [\alpha]$ zeros in the interval $(0, 1)$. Hence the theory of multipoint Padé approximants is applicable, and it gives us rather precise information about the structure of the numerators and denominators of r_{mn}^* (cf. [GoLa; StTo, §§6.1, 6.2]).

In the next step the error function e_n and the approximant r_{mn}^* will be transformed in such a way that the resulting function Ψ_n is analytic in $\mathbb{C} \setminus \mathbb{R}$ with possible exceptions in a disc $\Delta(R)$ with radius $R > 0$ around the origin. The function Ψ_n has boundary values from both sides of $\mathbb{R} \setminus \overline{\Delta(R)}$ that allow a comparison with a special logarithmic potential. The potential will be introduced after Theorem 2 below.

The transformation of e_n and r_{mn}^* into the function Ψ_n is carried out in several steps. The intermediate functions are defined as

$$(3.3) \quad r_n(z) := \frac{z^\alpha - r_{mn}^*(z)}{z^\alpha + r_{mn}^*(z)};$$

$$(3.4) \quad R_n(w) := \frac{4w^{2\alpha} - 1}{w^\alpha} r_n(\varepsilon_n^{1/\alpha} w) - \frac{1}{w^\alpha}, \quad \varepsilon_n := E_{mn}(x^\alpha, [0, 1]);$$

$$(3.5) \quad \Phi_n(w) := \frac{1}{8w^\alpha} \left(R_n(w) + \sqrt{R_n(w)^2 - 4} \right);$$

$$(3.6) \quad \Psi_n(w) := \begin{cases} \psi(\Phi_n(w)) & \text{for } \text{Im}(w) \geq 0, \\ \bar{\psi}(\Phi_n(w)) & \text{for } \text{Im}(w) < 0, \end{cases}$$

with

$$(3.7) \quad \psi(z) := \frac{z}{\sin \pi \alpha - i(\cos \pi \alpha)z}.$$

In (3.4) a new variable w is introduced implicitly by

$$(3.8) \quad w := \varepsilon_n^{-1/\alpha} z, \quad z \in \mathbb{C}.$$

The properties of each new function r_n, R_n, Φ_n have to be studied carefully. The properties of the last function Ψ_n is summarized in

Lemma 1. *The function Ψ_n is analytic and different from zero in $\mathbb{C} \setminus (\mathbb{R} \cup \overline{\Delta(R)})$, $R > 0$ appropriately chosen, and there exist constants c_1, \dots, c_4 such that*

$$(3.9) \quad |\log |\Psi_n(w)|| \leq c_1 |w|^{-2\alpha} \quad \text{for } w \in \mathbb{R}_- \setminus \Delta(R), n \geq n_0(c_1, R);$$

$$(3.10) \quad |\log |\Psi_n(w)w^\alpha 4 \sin \pi \alpha|| \leq c_2 |w|^{-\alpha} \quad \text{for } w \in \mathbb{R}_+ \setminus \Delta(R), n \geq n_0(c_2, R);$$

$$(3.11) \quad |\log |\Psi_n(w)|| \leq c_3 \quad \text{for } w \in \partial \Delta(R), n \geq n_0(c_3, R).$$

If we consider the representation

$$(3.12) \quad \log |\Psi_n(w)| = \psi_n(w) - \int g_D(w, t) d\mu_n(t),$$

where $D := \mathbb{C} \setminus (\mathbb{R}_- \cup \overline{\Delta(R)})$, $g_D(w, t)$ the Green function in D , ψ_n a harmonic function in D with

$$(3.13) \quad \psi_n(w) = \log |\Psi(w)| \quad \text{for } w \in \partial D,$$

and μ_n a measure with

$$(3.14) \quad \text{supp}(\mu_n) \subseteq [R, \infty] \quad \text{and} \quad \mu_n \geq 0 \quad \text{on } [R, \varepsilon_n^{-1/\alpha}],$$

then in addition to (3.9)–(3.11) we have the inequalities

$$(3.15) \quad \|\mu_n|_{[\varepsilon_n^{-1/\alpha}, \infty]}\| \leq 1,$$

$$(3.16) \quad |\mu_n([R, \varepsilon_n^{-1/\alpha}]) - 2n| \leq c_4 \quad \text{for } n \geq n_0(c_4, R).$$

Remark. Estimates (3.9), (3.10), and (3.16) contain the information that is most relevant for the proof of Theorem 1.

The function $\log |\Psi_n|$ will be compared with a special Green potential p_n . The definition of this potential is based on the following.

Theorem 2. *For each $a \in (1, \infty)$ there uniquely exists a Green potential*

$$(3.17) \quad p_a(w) := \int g_D(w, t) d\nu_a(t), \quad D := \mathbb{C} \setminus \mathbb{R}_-,$$

with

$$(3.18) \quad \nu_a \geq 0, \quad \text{supp}(\nu_a) = [b_a, a], \quad 0 < b_a < a,$$

that satisfies

$$(3.19) \quad p_a(w) \begin{cases} = \log w & \text{for } w \in [b_a, a], \\ > \log w & \text{for } w \in (0, b_a). \end{cases}$$

For the constant b_a and the measure ν_a appearing in (3.17)–(3.19) we have

$$(3.20) \quad b_a \rightarrow \sqrt{2}$$

and

$$(3.21) \quad \frac{1}{a} e^{\pi \sqrt{2 \|\nu_a\|}} \rightarrow 4 \quad \text{as } a \rightarrow \infty.$$

A proof of Theorem 2 can be derived from [St, Theorem 2]. It is necessary to change the domain of definition D by the transformation $w \mapsto 1/\sqrt{w}$. Basic tools in the proof of Theorem 2 are estimates for certain elliptical integrals.

The potential p_n for a comparison with the function $\log |\Psi_n|$ is now defined as

$$(3.22) \quad p_n(w) := -\alpha p_a(cw)$$

with

$$(3.23) \quad a := \left| \frac{4}{\varepsilon_n} \sin \pi \alpha \right|^{1/\alpha} \quad \text{and} \quad c := |4 \sin \pi \alpha|^{1/\alpha}.$$

From (3.21), (3.23), and (3.16) together with other information provided by Lemma 1 it then follows that

$$(3.24) \quad \left[\left| \frac{\varepsilon_n}{4 \sin \pi \alpha} \right|^{1/\alpha} e^{\pi \sqrt{2(2n)/\alpha}} \right]^\alpha = \frac{\varepsilon_n}{4 |\sin \pi \alpha|} e^{2\pi \sqrt{\alpha n}} \rightarrow 4^\alpha \quad \text{as } n \rightarrow \infty.$$

Theorem 1 then immediately follows from (3.24).

What we have given here is only a sketch of the overall structure of the proof of Theorem 1. Some of the steps demand rather subtle analysis.

Note Added in Proof. In [VC2] Varga and Carpenter have calculated numerical approximations for the right-hand side of (2.1) for the six values $\alpha = j/8$, $j = 1, 2, 3, 5, 6, 7$, and conjectured formula (2.1) on the basis of these numerical values. The numerical results for the two cases $\alpha = 1/4$ and $\alpha = 3/4$ are especially interesting since here the right-hand side of (2.1) is rational.

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